

Counterexamples in the theory of coerciveness for linear elliptic systems related to generalizations of Korn's second inequality

Patrizio Neff ^{*} and Waldemar Pompe [†]

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Abstract

We show that the following generalized version of Korn's second inequality with non-constant measurable matrix valued coefficients $P : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3}$

$$\|DuP + (DuP)^T\|_q + \|u\|_q \geq c \|Du\|_q \quad \text{for } u \in W_0^{1,q}(\Omega; \mathbb{R}^3), \quad 1 < q < \infty$$

is in general false, even if $P \in \text{SO}(3)$, while the Legendre-Hadamard condition and ellipticity on \mathbb{C}^n for the quadratic form $|DuP + (DuP)^T|^2$ is satisfied. Thus Gårding's inequality may be violated for formally positive quadratic forms.

Key words: Korn's second inequality, Gårding's inequality, coerciveness, elliptic systems, Legendre-Hadamard ellipticity condition

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1 Introduction

Gårding's inequality plays a crucial role in the theory of elliptic partial differential equations and systems of equations. In the case of systems related to the linear elasticity second order systems, with which we are concerned, Gårding's inequality gives sufficient conditions for weak coercivity. More precisely, let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let a mapping $A : \Omega \mapsto \text{Lin}(\mathbb{R}^{n \times n}, \mathbb{R}^{n \times n})$ be given. Define a bilinear form

$$a(u, v) := \int_{\Omega} \langle A(x).Du, Dv \rangle \, dx, \quad u, v \in C_0^\infty(\Omega, \mathbb{R}^n) \quad (1.1)$$

for simplicity without lower order terms. Here, $\langle X, Y \rangle := \sum_{i,j=1}^n X_{ij}Y_{ij}$ for $X, Y \in \mathbb{R}^{n \times n}$.

The problem is, under what set of assumptions on A does **weak coercivity** hold, i.e.

$$\exists \lambda, c > 0 \quad \forall u \in C_0^\infty(\Omega, \mathbb{R}^n) : \quad a(u, u) + \lambda \|u\|_2^2 \geq c \|Du\|_2^2. \quad (1.2)$$

This is a vector-valued form of Gårding's inequality [8]. Note that Gårding's inequality makes a statement about functions with compact support, only. By **strong coercivity** we mean an inequality of the type

$$\exists c > 0 \quad \forall u \in C_0^\infty(\Omega, \mathbb{R}^n) : \quad a(u, u) \geq c \|Du\|_2^2. \quad (1.3)$$

^{*}Corresponding author: Patrizio Neff, Lehrstuhl für Nichtlineare Analysis und Modellierung, Fakultät für Mathematik, Universität Duisburg-Essen, Campus Essen, Thea-Leymann Str. 9, 45127 Essen, Germany, email: patrizio.neff@uni-due.de, Phone +49 201 183 4243, Fax: +49 201 183 4394

[†]Waldemar Pompe, Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warszawa, Poland, email: pompe@mimuw.edu.pl. W. Pompe is supported by the Polish Ministry of Science grant no. N N201 397837 (years 2009-2012).

It is well known [30, p.74] that Gårding's inequality (1.2) is true, provided that A is uniformly continuous on Ω (continuous up to the boundary) and A satisfies a uniform Legendre-Hadamard condition

$$\exists c > 0 \forall \xi, \eta \in \mathbb{R}^n \setminus \{0\} : \langle A(x).(\xi \otimes \eta), \xi \otimes \eta \rangle \geq c |\xi|^2 |\eta|^2. \quad (1.4)$$

It is known that Caccioppoli's inequality, which is an integral inequality estimating the derivatives Du of weak solutions of the corresponding elliptic system in terms of u itself and which is decisive for showing regularity, may break down for merely measurable A satisfying the Legendre-Hadamard ellipticity condition, see [6], while it is true for uniformly continuous A . K. Zhang [31, 32] presented an example such that the mapping A has measurable coefficients, $\Omega \subset \mathbb{R}^3$, satisfying

$$\langle A(x).(\xi \otimes \eta), \xi \otimes \eta \rangle = |\xi|^2 |\eta|^2, \quad (1.5)$$

but

$$\forall \lambda > 0 \exists u \in C_0^\infty(\Omega, \mathbb{R}^3) : \int_{\Omega} \langle A(x).Du, Du \rangle + \lambda |u|^2 dx < 0. \quad (1.6)$$

However, this negative answer for weak coercivity of the bilinear form a for A with measurable coefficients is using a quadratic form a which is not formally positive. By **formal positivity** we understand that there exists some mapping A (which then corresponds to the square-root of A from (1.6)) such that a can be written as

$$a(u, u) = \int_{\Omega} |A(x).Du|^2 dx \geq 0. \quad (1.7)$$

For further use let us define for a given continuous mapping $\hat{A} : \mathbb{R}^{n \times n} \mapsto \text{Lin}(\mathbb{R}^{n \times n}, \mathbb{R}^{n \times n})$

$$a_{\Phi}(u, v) = \int_{\Omega} \langle \hat{A}(\Phi(x)).Du, Dv \rangle dx, \quad u, v \in C_0^\infty(\Omega, \mathbb{R}^n), \quad (1.8)$$

where $\Phi \in L^\infty(\Omega, \mathbb{R}^{n \times n})$, at least. For such a structure we have [14, Th. 6.5.1, p.253]

$$\exists \lambda, c > 0 \quad \forall u \in H_0^1(\Omega) : \quad a_{D\varphi}(u, u) + \lambda(\varphi) \|u\|_2^2 \geq c \|Du\|_2^2 \quad (1.9)$$

if $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^n)$ and $A(D\varphi(x))$ satisfies the uniform Legendre-Hadamard ellipticity condition. If the bilinear form is formally positive, has variational structure $a(u, u) = \int_{\Omega} |\hat{A}(x).Du|^2 dx$ and satisfies the additional ellipticity condition

$$\tilde{A}(x) \in \text{Lin}(\mathbb{C}^{n \times n}, \mathbb{C}^{n \times n}), \quad (1.10)$$

$$\tilde{A}(x).(\xi \otimes \eta) \neq 0 \quad \text{whenever } \xi, \eta \in \mathbb{C}^n \text{ and } \xi \neq 0, \eta \neq 0,$$

which implies the Legendre-Hadamard ellipticity condition, then one has the stronger inequality for functions without boundary conditions

$$\exists \lambda, c > 0 \forall u \in H^1(\Omega) : \quad a(u, u) + \lambda \|u\|_2^2 \geq c \|Du\|_2^2 \quad (1.11)$$

if \tilde{A} is uniformly continuous on $\overline{\Omega}$, see [9, 10].

We are especially interested in such quadratic form that arise in generalizations of Korn's inequality [4, 26], namely we consider the formally positive bilinear form

$$a_P(u, v) := \int_{\Omega} \langle \text{sym}(Du P), \text{sym}(Dv P) \rangle dx = \int_{\Omega} \langle [\text{sym}(Du P)] P^T, Dv \rangle dx \quad (1.12)$$

for given $P \in L^\infty(\Omega, \text{GL}^+(3))$. This defines the operator \hat{A} from above via $\hat{A}(P).X := [\text{sym}(XP)] P^T$. Here, $\text{sym } X := \frac{1}{2}(X + X^T)$. Bilinear forms having this nonstandard shape are nonetheless ubiquitous, they appear e.g. in micromorphic elasticity models [11, 23, 20], in geometrically exact formulations of plasticity [19, 18], in Cosserat models [21] or in thin shell models [1, 2, 22].

Note that a_P cannot be reduced to a quadratic form of the linearized elastic strains $\text{sym } Du$ for general P . Further discussions of coercivity for quadratic forms defined on the linearized strains and lack of coercivity for such models with inhomogeneous material parameters but satisfying the Legendre-Hadamard ellipticity condition can be found in [5, 33, 34, 35].

1.1 The geometrically exact Cosserat model

In order to see the significance of the new bilinear form (1.12) we briefly introduce the variational **geometrically exact Cosserat model**: the goal in this extended continuum model is to find the deformation $\varphi : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ and the Cosserat microrotation $\bar{R} : \Omega \subset \mathbb{R}^3 \mapsto \text{SO}(3)$

$$\begin{aligned} \int_{\Omega} W(D\varphi, \bar{R}) + W_{\text{curv}}(D\bar{R}) - \langle f, \varphi \rangle \, dx &\mapsto \min. (\varphi, \bar{R}), \\ W(D\varphi, \bar{R}) &= \mu_e |\text{sym}(\bar{R}^T D\varphi - \mathbb{1})|^2 + \frac{\lambda_e}{2} \text{tr} [\text{sym}(\bar{R}^T D\varphi - \mathbb{1})]^2, \\ W_{\text{curv}}(D\bar{R}) &= \mu_e \left(\frac{L_c^2}{2} |\text{Curl } \bar{R}|^2 + \frac{L_c^q}{q} |\text{Curl } \bar{R}|^q \right). \end{aligned} \quad (1.13)$$

Here, $L_c > 0$ is defining an intrinsic length scale in the model, while the elastic Lamé coefficients satisfy $\mu_e, 3\lambda_e + 2\mu_e > 0$. The model is geometrically exact in the sense that it is invariant under the rigid rotation $(\varphi, \bar{R}) \mapsto (Q\varphi, Q\bar{R})$ for any constant $Q \in \text{SO}(3)$. This sets the model apart from linear elasticity. Existence for this model hinges on the coerciveness properties of

$$\int_{\Omega} |\bar{R}^T D\varphi + D\varphi^T \bar{R}|^2 \, dx = \int_{\Omega} |D\varphi \bar{R}^T + \bar{R} D\varphi^T|^2 \, dx = 4 a_{\bar{R}^T}(\varphi, \varphi) \quad (1.14)$$

at given rotation tensor $\bar{R} \in \text{SO}(3)$. Our result below shows that coercivity of the Cosserat model w.r.t. deformations φ needs some additional smoothness which can be ensured via the curvature contribution W_{curv} , see [24]. For applications of the Cosserat model in materials science, we refer to [15, 16, 25].

For $P = \mathbb{1}$ we obtain

$$a_{\mathbb{1}}(u, u) = \int_{\Omega} \langle \text{sym } Du, \text{sym } Du \rangle \, dx, \quad (1.15)$$

which is a measure for the linear elastic strain. Gårding's inequality is then nothing else but a simplified version of Korn's second inequality [7, 27] on $H_0^1(\Omega)$, i.e.

$$\int_{\Omega} |\text{sym } Du|^2 + |u|^2 \, dx \geq c \|Du\|_2^2. \quad (1.16)$$

Since the bilinear form a_P satisfies a uniform Legendre-Hadamard ellipticity condition

$$\begin{aligned} \langle \text{sym}((\xi \otimes \eta) P), \text{sym}((\xi \otimes \eta) P) \rangle &= \langle \text{sym}(\xi \otimes P^T \eta), \text{sym}(\xi \otimes P^T \eta) \rangle \\ &\geq \frac{1}{2} |\xi|^2 |P^T \eta|^2 \geq \frac{1}{2} \lambda_{\min}(PP^T) |\xi|^2 |\eta|^2 \end{aligned} \quad (1.17)$$

for P such that $\det[P] \geq \mu > 0$ and $P \in C(\bar{\Omega}, \mathbb{R}^{n \times n})$, we infer weak coercivity e.g. from [30, p.74], i.e.

$$\exists \lambda, c > 0 \, \forall u \in H_0^1(\Omega) : \quad a_P(u, u) + \lambda \|u\|_2^2 \geq c \|Du\|_2^2. \quad (1.18)$$

Due to the special structure of the bilinear form a_P it is easy to see that the ellipticity condition (1.10) is also satisfied and we know furthermore that a_P is strictly coercive, provided that $\det[P] \geq \mu > 0$ and $P \in C(\bar{\Omega}, \mathbb{R}^{n \times n})$, see [12, 13, 17, 28]. If P is invertible but merely measurable, then we know that strong coercivity, i.e. Korn's first inequality, is in general not true [29]. If P is invertible, measurable, symmetric and positive definite, then strict coercivity in $H_0^1(\Omega)$ is obtained, without further smoothness assumptions [28]. Finally, if $P^{-1} = D\varphi$ for a diffeomorphism $\varphi \in C(\bar{\Omega}, \mathbb{R}^n)$ ($D\varphi \in L^\infty$) then strict coercivity is obtained as well by a simple transformation of variables argument.

By and large, a_P from (1.12) is not strictly coercive if P is only invertible and measurable. Nevertheless, weak coercivity for a_P , i.e., the generalization of Korn's second inequality, could still be true.

However, in this contribution we show by way of counterexamples that weak coercivity

$$\exists \lambda, c > 0 \, \forall u \in H_0^1(\Omega) : \quad a_P(u, u) + \lambda \|u\|_2^2 \geq c \|Du\|_2^2 \quad (1.19)$$

fails in general for P invertible and measurable. We generalize our counterexamples in the obvious way to the L^q -setting, i.e.

$$\exists \lambda, c > 0 \, \forall u \in W_0^{1,q}(\Omega, \mathbb{R}^n) : \quad \int_{\Omega} |\text{sym } Du P|^q + \lambda |u|^q \, dx \geq c \|Du\|_q^q \quad (1.20)$$

does not hold for $q > 1$.

2 Main Part

For simplicity, we restrict ourselves to three space dimensions. From now on let Ω be an open, bounded set in \mathbb{R}^3 and let $P: \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3}$ with $P \in L^\infty(\Omega)$ and $\det[P(x)] \geq \mu > 0$ be given. Assume moreover $q > 1$.

Inequality (1.20) is equivalent to

$$\exists c > 0 \quad \forall u \in W_0^{1,q}(\Omega; \mathbb{R}^3) : \quad \|DuP + (DuP)^T\|_q + \|u\|_q \geq c\|Du\|_q. \quad (2.1)$$

It is known that inequality (2.1) holds, if we additionally assume that $P \in C(\overline{\Omega})$ [9, 10]. This is a generalization of Korn's second inequality. However, we shall show that this inequality is not in general true with noncontinuous, bounded coefficients P , under the structural assumption $P \in SO(3)$. We even show that the following weaker inequality is not valid in this case:

$$\|DuP + (DuP)^T\|_q + \|u\|_\infty \geq c\|Du\|_q \quad \text{for } u \in W_0^{1,q}(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3). \quad (2.2)$$

We present two counterexamples to inequality (2.2). The first one assumes that the coefficients $P(x)$ are bounded and satisfy $\det[P(x)] = 1$ a.e. on Ω , while the second counterexample assumes more about the coefficients: $P \in SO(3)$. The construction in the second case is based on the following result by A. Cellina and S. Perrotta [3]: *If Ω is an open, bounded set in \mathbb{R}^3 , then there exists a mapping $u \in W_0^{1,\infty}(\Omega; \mathbb{R}^3)$ such that $Du(x) \in O(3)$ a.e. on Ω .*

Even if the counterexample in the second case provides also a counterexample in the first one, we present a more elementary construction in the first case, which does not require the strong result of A. Cellina and S. Perrotta [3]. The ideas of our constructions are similar to the constructions presented by the second author in [28, 29].

Our method of construction is direct and yields P having a finite number of elements. This result - due to a complicated structure of the rank-one connections - would be hard to obtain by the convex integration method.

To deal better with constants, we use the following definition of the L^q -norm of a mapping $P: \Omega \rightarrow \mathbb{R}^{3 \times 3}$. For

$$P(x) = \begin{pmatrix} p_1(x) \\ p_2(x) \\ p_3(x) \end{pmatrix} \quad (x \in \Omega)$$

define

$$\|P\|_q^q = \int_{\Omega} (|p_1(x)|^q + |p_2(x)|^q + |p_3(x)|^q) dx,$$

where $|p_i|$ denotes the Euclidean norm of the vector $p_i \in \mathbb{R}^3$.

Theorem 1.

For each $q > 1$ and any open, bounded subset Ω of \mathbb{R}^3 , there exist $P \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$ with $\det[P](x) = 1$ and a sequence $u_n \in W_0^{1,q}(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$, such that:

- (a) $Du_n P + (Du_n P)^T = 0$ on the set Ω ,
- (b) $\|Du_n\|_q = 2^{1/q}$,
- (c) $\|u_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$.

Proof

Let $\Omega_1, \Omega_2, \dots$ be open, disjoint subsets of the set Ω , such that the set $\Omega \setminus (\Omega_1 \cup \Omega_2 \cup \dots)$ has the measure zero. Let moreover R be a fixed rotation in \mathbb{R}^3 such that $Re_i \neq \pm e_j$ for all $i, j \in \{1, 2, 3\}$. On each of the sets Ω_n we construct two Vitali coverings: one of them with cubes Q_{ni} ($i = 1, 2, \dots$), whose edges are parallel to the vectors e_1, e_2 and e_3 and the other with cubes S_{nj} ($j = 1, 2, \dots$), whose edges are parallel to the vectors Re_1, Re_2 and Re_3 .

Therefore we have $\Omega_n = Q_{n1} \cup Q_{n2} \cup \dots$, where the interiors of the cubes Q_{ni} ($i = 1, 2, \dots$) are disjoint, and similarly $\Omega_n = S_{n1} \cup S_{n2} \cup \dots$, where the interiors of the cubes S_{nj} ($j = 1, 2, \dots$) are disjoint. We may moreover assume that the length of the edge of each cube Q_{ni} and S_{nj} is

at most $\frac{2|\Omega_n|^{1/q}}{n}$.

Now on each of the sets Ω_n define two mappings: $u_n^1, u_n^2: \Omega_n \rightarrow \mathbb{R}$ as follows:

$$u_n^1(x) = \text{dist}(x, \partial Q_{ni}) \quad \text{for } x \in Q_{ni} \quad (i = 1, 2, \dots)$$

and

$$u_n^2(x) = \text{dist}(x, \partial S_{nj}) \quad \text{for } x \in S_{nj} \quad (j = 1, 2, \dots).$$

Then $u_n^1, u_n^2 \in W_0^{1,\infty}(\Omega_n)$ and $|Du_n^1(x)| = |Du_n^2(x)| = 1$ a.e. on Ω_n . Moreover, we have

$$(*) \quad |u_n^1(x)| \leq \frac{|\Omega_n|^{1/q}}{n} \quad \text{and} \quad |u_n^2(x)| \leq \frac{|\Omega_n|^{1/q}}{n} \quad (x \in \Omega_n).$$

Since $Re_i \neq \pm e_j$, the vectors $Du_n^1(x), Du_n^2(x)$ are not parallel and therefore they span a 2-dimensional plane $\pi(x)$ in \mathbb{R}^3 . Let $v_n(x)$ be the vector orthogonal to this plane, such that

$$\det \begin{pmatrix} -Du_n^2(x) \\ Du_n^1(x) \\ v_n(x) \end{pmatrix} = 1 \quad (x \in \Omega_n).$$

Then since $Re_i \neq \pm e_j$, we obtain $c_1 < |v_n^3(x)| < c_2$, where the constants c_1 and c_2 are positive and depend only on R .

Define

$$P_n(x) = \begin{pmatrix} -Du_n^2(x) \\ Du_n^1(x) \\ v_n(x) \end{pmatrix}^{-1} \quad \text{for } x \in \Omega_n$$

and $P_n(x) = 0$ for $x \in \Omega \setminus \Omega_n$. Then $P_n \in L^\infty(\Omega_n; \mathbb{R}^{3 \times 3})$. Finally let

$$P(x) = \sum_{n=1}^{\infty} P_n(x).$$

Since the supports of the mappings $P_n(x)$ are disjoint, the above sum is actually a single summand for almost each $x \in \Omega$ and therefore $P(x) = P_n(x)$ for a.e. $x \in \Omega_n$. It follows therefore that $P \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$. It is also clear that $\det[P(x)] = 1$.

Now define the mappings $u_n: \Omega \rightarrow \mathbb{R}^3$ with

$$u_n(x) = \frac{1}{|\Omega_n|^{1/q}} (u_n^1(x), u_n^2(x), 0) \quad \text{for } x \in \Omega_n,$$

and $u_n(x) = 0$ for $x \in \Omega \setminus \Omega_n$. Then using (*), we obtain

$$|u_n(x)| \leq \frac{1}{|\Omega_n|^{1/q}} \cdot (|u_n^1(x)| + |u_n^2(x)|) \leq \frac{2}{n} \quad (x \in \Omega).$$

Hence $u_n \in W_0^{1,\infty}(\Omega)$ and the property (c) holds. Moreover, for $x \in \Omega_n$ we have

$$Du_n(x)P(x) = Du_n(x)P_n(x) = \begin{pmatrix} 0 & |\Omega_n|^{-1/q} & 0 \\ -|\Omega_n|^{-1/q} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $Du_n(x)P(x) = 0$ for $x \in \Omega \setminus \Omega_n$. This shows property (a). Finally, to see property (b) note that

$$\|Du_n\|_q^q = \frac{1}{|\Omega_n|} \int_{\Omega_n} (|Du_n^1(x)|^q + |Du_n^2(x)|^q) dx = 2,$$

and the conclusion (b) follows.

Theorem 2.

For each $q > 1$ and any open, bounded subset Ω of \mathbb{R}^3 , there exist a mapping $P: \Omega \rightarrow \mathbb{R}^{3 \times 3}$ with $P(x) \in SO(3)$ and a sequence $u_n \in W_0^{1,q}(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$, such that:

- (a) $Du_n P + (Du_n P)^T = 0$ for $x \in \Omega$,
- (b) $\|Du_n\|_q = 2^{1/q}$,
- (c) $\|u_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$.

Proof

The construction is based on the following result of A. Cellina and S. Perrotta [3]: *If Ω is an open, bounded set in \mathbb{R}^3 , then there exists a mapping $u \in W_0^{1,\infty}(\Omega; \mathbb{R}^3)$ such that $Du(x) \in O(3)$ a.e. on Ω .*

Let Ω be represented, up to a set of measure 0, by a union of disjoint open sets $\Omega_1, \Omega_2, \dots$. On each of the sets Ω_n we construct a Vitali covering with cubes Q_{ni} ($i = 1, 2, \dots$). Therefore we have $\Omega_n = Q_{n1} \cup Q_{n2} \cup \dots$, where the interiors of the cubes Q_{ni} ($i = 1, 2, \dots$) are disjoint. We may moreover assume that the length of the edge of each cube Q_{ni} ($i = 1, 2, \dots$) is at most $\frac{1}{n} |\Omega_n|^{1/q}$.

Let Q be a unit cube and let $v \in W_0^{1,\infty}(Q; \mathbb{R}^3)$ such that $Dv(x) \in O(3)$ a.e. on Q . Let $u(x) = (v_1(x), v_2(x), 0)$. Then $Du_1(x), Du_2(x)$ have the length 1 and are orthogonal for a.e. $x \in Q$.

Set $c = \|u\|_\infty$. Then scaling and translating the mapping u , we construct on each cube Q_{ni} a mapping $u_{ni} \in W_0^{1,\infty}(Q_{ni}; \mathbb{R}^3)$ with

$$Du_{ni}(x) = \begin{pmatrix} Du_{ni}^1(x) \\ Du_{ni}^2(x) \\ 0 \end{pmatrix} \quad \text{for a.e. } x \in Q_{ni},$$

where $Du_{ni}^1(x)$ and $Du_{ni}^2(x)$ have length 1 and are orthogonal for a.e. $x \in Q_{ni}$ and

$$|u_{ni}(x)| \leq c \cdot \frac{|\Omega_n|^{1/q}}{n} \quad (x \in Q_{ni}).$$

Extend the mappings u_{ni} ($i = 1, 2, \dots$) from Q_{ni} to Ω by setting $u_{ni}(x) = 0$ on $\Omega \setminus Q_{ni}$ and define

$$u_n(x) = \frac{1}{|\Omega_n|^{1/q}} \sum_{i=1}^{\infty} u_{ni}(x) \quad (x \in \Omega).$$

Then for a.e. $x \in Q_{ni}$ we have

$$u_n(x) = \frac{1}{|\Omega_n|^{1/q}} u_{ni}(x).$$

Thus

$$|u_n(x)| = \frac{1}{|\Omega_n|^{1/q}} |u_{ni}(x)| \leq \frac{c}{n} \quad (x \in \Omega).$$

This shows the property (c).

Now define $P(x)$ as follows. If $x \in Q_{ni}$, then define $P(x)$ such that $P(x)^T$ is the rotation, which takes the vectors $Du_{ni}^1(x), Du_{ni}^2(x)$ to the vectors $(0, 1, 0), (-1, 0, 0)$, respectively (such a rotation exists, since the vectors $Du_{ni}^1(x), Du_{ni}^2(x)$ have the length 1 and are orthogonal). Then we have

$$Du_n(x)P(x) = \begin{pmatrix} 0 & |\Omega_n|^{-1/q} & 0 \\ -|\Omega_n|^{-1/q} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for } x \in \Omega_n,$$

and $Du_n(x)P(x) = 0$ for $x \in \Omega \setminus \Omega_n$. This shows property (a). Finally, to see property (b) note that

$$\|Du_n\|_q^q = \int_{\Omega_n} (|Du_n^1(x)|^q + |Du_n^2(x)|^q) dx = \int_{\Omega_n} \frac{2}{|\Omega_n|} dx = 2,$$

and the conclusion (b) follows.

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